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Baxter-Bazhanov Model, Frenkel-Moore Equation and the Braid Group

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Abstract

In this paper the three-dimensional vertex model is given, which is the duality of the three-dimensional Baxter-Bazhanov (BB) model. The braid group corresponding to Frenkel-Moore equation is constructed and the transformations R, I are found. These maps act on the group and denote the rotations of the braids through the angles π about some special axes. The weight function of another three-dimensional vertex model related the 3D lattice integrable model proposed by Boos, Mangazeev, Sergeev and Stroganov is presented also, which can be interpreted as the deformation of the vertex model corresponding to the BB model.

Short title: BB Model and Braid Group

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1 Introduction

The braid group has a deep connection with the Yang-Baxter equation which plays an important role in the exactly solvable model in two dimensions in statistical mechanics. The tetrahedron equation is an integrable condition of statistical model in three-dimensions due to which gives the commutativity of the layer to layer transfer matrixes [1]. The Frenkel-Moore version of the Tetrahedral equation [2, 3] is formulated as:

$$S_{123}S_{124}S_{134}S_{234} = S_{234}S_{134}S_{124}S_{123} \tag{1}$$

where S∈End($V^{\otimes 3}$) and each side of the equation acts on $V^{\otimes 4} = V_1V_2V_3V_4$, and S_{123}, for example, acts on V_4 identically. Then we can ask a question that what is the braid group corresponding to Frenkel-Moore equation. An answer will be given in this paper.

Just as the Yang-Baxter equation can describe the scattering in two dimensions, with a factorizable matrix, the tetrahedron equation gives the relations among the scattering amplitudes of three strings [4]. And the three labeling schemes exist [5]. With the cell (or vacuum) labeling Bazhanov and Baxter generalized the two-state Zamolodchikov model [4] to an arbitrary number of state [6]. Then the three-dimensional star-star relation is proved [7, 8, 9] and the star-square relation is discussed [1, 10]. Recently Mangazeev and Boos *etal* obtained the solution of modified tetrahedron equation [11, 12] in terms of elliptic functions which generalized the result of ref. [13]. Frenkel-Moore equation can also be gotten by using string labeling. So we discuss the braid relation of Frenkel-Moore equation in the following section by regarding the three-string cross as the "elementary" braid [14].

The vertex type tetrahedron equations were discussed in refs. [15, 5] and the discrete symmetry groups of vertex models were studied by Boukraa *et al* [16]. Mangazeev *et al* [17] proposed a three-dimensional vertex model and the weight function of this model can be obtained from Baxter-Bazhanov model [18] when taking some limits. The three-dimensional vertex model corresponding to Baxter-Bazhanov

model is constructed in this paper. And the six spectrums with a constrained condition relate to the six spaces which the vertex type tetrahedron equation is defined in.

This paper is organized as the follows. In section 2, the three-dimensional vertex model is given, which is a duality of the three-dimensional Baxter-Bazhanov model. The braid group corresponding to the Frenkel-Moore equation is constructed in section 3. In section 4, the transformations R, I are discussed. Finally some conclusions are given and Boltzmann weight of another three-dimensional vertex model is presented. It corresponds to the 3D lattice integrable models proposed by Boos *et al* [11, 12] and can be regarded as the deformation of the vertex model corresponding to BB model.

2 The Three-Dimensional Vertex Model

The Baxter-Bazhanov model is an Interaction-Round-a-Cube (IRC) Model. The weight function of it has the form [7]

$$
\begin{aligned}
&W_P(a|efg|bcd|h) \\
&\quad = \frac{\omega^{fb}}{\omega^{ag}}\left[\frac{w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24}|a+d, e+f)}{w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24}|g+h, c+b)}\right]^{1/2} \\
&\quad \times \left[\frac{w(x_4, x_{34}, x_3|e+h, d+c)}{w(x_4, x_{34}, x_3|a+b, f+g)}\right]^{1/2} \\
&\quad \times \left[\frac{w(x_2, x_{12}, x_1|e+g, a+c)}{w(x_2, x_{12}, x_1|d+b, f+h)}\right]^{1/2} \frac{\omega^{(ag+gb+bh)/2}}{\omega^{(hd+de+ea)/2}} \\
&\quad \times \left\{\sum_{\sigma\in Z_N} \frac{w(x_3, x_{13}, x_1|d, h+\sigma)w(x_4, x_{24}, x_2|a, g+\sigma)}{w(x_4, x_{14}, x_1|e, c+\sigma)w(x_3/\omega, x_{23}, x_2|f, b+\sigma)}\right\}_0 .
\end{aligned} \tag{2}
$$

where the subscript "0" after the curly brackets indicates that the expression in the braces is divided by itself with the zero exterior spins and we have used the notations

$$
w(x,y,z|k,l) = w(x,y,z|k-l)\Phi(l), \quad w(x,y,z|l) = \prod_{j=1}^{l} \frac{y}{z - x\omega^j}, \quad k, l \in Z_N, \tag{3}
$$

$$x^N + y^N = z^N,\ \Phi(l) = \omega^{l(l+N)/2},\ \omega^{1/2} = \exp(\pi i/N),\ x_i^N - x_j^N = x_{ij}^N, \tag{4}$$

for $i < j$ and $i, j = 1, 2, 3, 4$. It satisfies the cube type tetrahedron equation

$$\begin{aligned}
&\sum_d W(a_4|c_2c_1c_3|b_1b_3b_2|d)W'(c_1|b_2a_3b_1|c_4dc_6|b_4) \\
&\quad \times W''(b_1|dc_4c_3|a_2b_3b_4|c_5)W'''(d|b_2b_4b_3|c_5c_2c_6|a_1) \\
&\qquad = \sum_d W'''(b_1|c_1c_4c_3|a_2a_4a_3|d)W''(c_1|b_2a_3a_4|dc_2c_6|a_1) \\
&\qquad\quad \times W'(a_4|c_2dc_3|a_2b_3a_1|c_5)W(d|a_1a_3a_2|c_4c_5c_6|b_4)
\end{aligned} \tag{5}$$

where W, W', W'' and W''' are some four sets of Boltzmann weights. By using the symmetry properties of the Boltzmann weights and setting

$$u = \frac{x_1}{\omega x_2},\ v = \frac{x_4}{x_3},\ z = \frac{z_1}{z_2},\ z_1 = \frac{x_{13}}{\omega x_{14}},\ z_2 = \frac{x_{23}}{x_{24}}, \tag{6}$$

the Boltzmann weight of the Baxter-Bazhanov model can be written into the vertex form

$$\begin{aligned}
R(u,z,v)_{i_1i_2i_3}^{j_1j_2j_3} &= (-)^{j_2}(\omega^{1/2})^{j_1j_2+j_2j_3+j_1j_3}\left[\frac{w(u,j_1)w(z_2/(\omega z_1),-i_2)w(v,i_3)}{w(u,i_1)w(z_2/(\omega z_1),-j_2)w(v,j_3)}\right]^{1/2} \\
&\times\left\{\sum_{\sigma\in Z_N}\frac{w(\omega v z_1,\sigma+j_2+j_3)w(z_2,\sigma)s(\sigma,j_1)}{w(z_1,\sigma+j_2)w(vz_2,\sigma+i_3)}\right\}_0
\end{aligned} \tag{7}$$

with

$$\frac{w(v,a)}{w(v,0)} = [\Delta(v)]^a \prod_{j=1}^{a}(1-\omega^j v)^{-1},\quad \Delta(v) = (1-v^N)^{1/N} \tag{8}$$

where $s(a,b) = \omega^{ab}$ and the spin variables $i_1,\ i_2,\ i_3,\ j_1,\ j_2,\ j_3$ satisfy the conditions $i_1 + i_2 = j_1 + j_2,\ i_2 + i_3 = j_2 + j_3$. The Boltzmann weights satisfy the vertex type tetrahedron equation

$$\begin{aligned}
&\sum_{\substack{\{k_i\},\\ i=1,\cdots,6}} R(u_1,u_2,u_3)_{i_1,i_2,i_3}^{k_1,k_2,k_3} R(u_1,u_4,u_5)_{k_1i_4i_5}^{j_1k_4k_5} R(u_2,u_4,u_6)_{k_2k_4i_6}^{j_2j_4k_6} R(u_3,u_5,u_6)_{k_3k_5k_6}^{j_3j_5j_6} = \\
&\sum_{\substack{\{k_i\},\\ i=1,\cdots,6}} R(u_3,u_5,u_6)_{i_3,i_5,i_6}^{k_3,k_5,k_6} R(u_2,u_4,u_6)_{i_2i_4k_6}^{k_2k_4j_6} R(u_1,u_4,u_5)_{i_1k_4k_5}^{k_1j_4j_5} R(u_1,u_2,u_3)_{k_1k_2k_3}^{j_1j_2j_3}
\end{aligned} \tag{9}$$

where

$$u_1=\frac{x_1}{\omega x_2}=\frac{x'_1}{\omega x'_2},\qquad u_2=\frac{x_{13}x_{24}}{\omega x_{14}x_{23}}=\frac{x''_1}{\omega x''_2},\quad u_3=\frac{x_4}{x_3}=\frac{x'''_1}{\omega x'''_2},$$
$$u_4=\frac{x'_{13}x'_{24}}{\omega x'_{14}x'_{23}}=\frac{x''_{13}x''_{24}}{\omega x''_{14}x''_{23}},\quad u_5=\frac{x'_4}{x'_3}=\frac{x'''_{13}x'''_{24}}{\omega x'''_{14}x'''_{23}},\quad u_6=\frac{x''_4}{x''_3}=\frac{x'''_4}{x'''_3}. \tag{10}$$

In this way, we get a three-dimensional vertex model [19] which corresponds to the Baxter-Bazhanov model. The spectrums $u_i (i=1,2,\cdots,6)$ appeared in the above tetrahedron equation satisfy the condition

$$\begin{aligned}&\Big[sin\frac{\theta_1+\theta_2+\theta_3}{2}sin\frac{-\theta_1+\theta_2+\theta_3}{2}sin\frac{-\theta_3+\theta_5+\theta_6}{2}sin\frac{\theta_3+\theta_5-\theta_6}{2}\Big]^{1/2}\\ &-\Big[sin\frac{\theta_1-\theta_2+\theta_3}{2}sin\frac{\theta_1+\theta_2-\theta_3}{2}sin\frac{\theta_3-\theta_5+\theta_6}{2}sin\frac{\theta_3+\theta_5+\theta_6}{2}\Big]^{1/2}\\ &=sin\theta_3\Big[sin\frac{\theta_2+\theta_4-\theta_6}{2}sin\frac{-\theta_2+\theta_4+\theta_6}{2}\Big]^{1/2}\end{aligned} \tag{11}$$

where we have parametrized the spectrums of the Boltzmann weights as

$$u_i=\omega^{-1/2}[ctg(\frac{\theta_i}{2})]^{2/N},\qquad i=1,2,\cdots,6. \tag{12}$$

3 Braid Group $\hat{\mathbf{B}}_N$

The shorthand notation of the vertex type tetrahedron equation (9) is

$$R_{123}R_{145}R_{246}R_{356}=R_{356}R_{246}R_{145}R_{123}, \tag{13}$$

which can be reformulated as [15]

$$R_{12,13,23}R_{12,14,24}R_{13,14,34}R_{23,24,34}=R_{23,24,34}R_{13,14,34}R_{12,14,24}R_{12,13,23} \tag{14}$$

by using the following translation [5]:

$$1\to 12,\quad 2\to 13,\quad 3\to 23,\ 4\to 14,\quad 5\to 24,\quad 6\to 34. \tag{15}$$

Then we can wright down the Frenkel-Moore equation (1) (see ref.[5]). The relations of the braid group corresponding to the Frenkel-Moore equation are discussed as

follows. For $N+1$ strings, let us express the "elementary" braid α_i and $\alpha_i^{-1}(i=1,2,\cdots,N-1)$ by Fig.1 and Fig.2 respectively. In Fig.1, $i+1$-string is on the bottom, $i-1$-string is on the top and i-string is between the $i-1$-string and $i+1$-string. In Fig.2, $i-1$-string is on the bottom, $i+1$-string is on the top and i-string is also between the $i-1$-string and $i+1$-string. When we define the product of α_i, and $\alpha_j(1\leq i,j\leq N-1)$ as that α_j acts on $N+1$ strings on which α_i has acted, $\alpha_i,\alpha_i^{-1},\alpha_j,\alpha_j^{-1},\cdots$ and all of the arbitrary products of them form a group $\hat{\mathrm{B}}_N$ in which the identity element is the no crossed $N+1$ strings. The "elementary" braid α_i satisfy the following relations:

$$\alpha_i\alpha_j=\alpha_j\alpha_i,\quad |i-j|\geq 3, \tag{16}$$

$$\alpha_i\alpha_{i\pm1}\alpha_i\alpha_{i\pm1}=\alpha_{i\pm1}\alpha_i\alpha_{i\pm1}\alpha_i, \tag{17}$$

$$\alpha_{i\pm1}\alpha_{i\mp1}\alpha_{i\pm1}\alpha_i=\alpha_i\alpha_{i\mp1}\alpha_{i\pm1}\alpha_{i\mp1}, \tag{18}$$

$$\alpha_{i\pm1}\alpha_{i\mp1}\alpha_i\alpha_{i\pm1}=\alpha_{i\mp1}\alpha_i\alpha_{i\pm1}\alpha_{i\mp1}, \tag{19}$$

$$\alpha_i\alpha_i^{-1}=\alpha_i^{-1}\alpha_i=E. \tag{20}$$

Notice that there are two relations in equations (17), (18) and (19), respectively. The first ones of them can be proved graphically by Fig.3, Fig.4 and Fig.5. The second relations of them can be proved similarly. From these relations it can be gotten easily that

$$\alpha_{i\pm1}\alpha_{i\mp1}(\alpha_{i\pm1}\alpha_i)^{2n+1}=(\alpha_i\alpha_{i\mp1})^{2n+1}\alpha_{i\pm1}\alpha_{i\mp1}, \tag{21}$$

$$\alpha_{i\pm1}\alpha_{i\mp1}(\alpha_i\alpha_{i\pm1})^{2m+1}=(\alpha_{i\mp1}\alpha_i)^{2m+1}\alpha_{i\pm1}\alpha_{i\mp1}, \tag{22}$$

$$\alpha_{i\pm1}\alpha_{i\mp1}(\alpha_i\alpha_{i\pm1})^{2l}=(\alpha_i\alpha_{i\mp1})^{2l}\alpha_{i\pm1}\alpha_{i\mp1}, \tag{23}$$

where m,n,l, are all the arbitrary integers.

Now we discuss the generators of the braid group $\hat{\mathrm{B}}_N(N\geq 2)$ as follows. Firstly , $\hat{\mathrm{B}}_2$ and $\hat{\mathrm{B}}_3$ have two and four generators respectively. And $\hat{\mathrm{B}}_4$ has also four generators owing to $\alpha_2=\alpha_1^{-1}\beta^2\alpha_1^{-1}\beta^{-1}$ and $\alpha_3=\beta\alpha_1\beta^{-1}$ where $\beta=\alpha_1\alpha_2\alpha_3$. For $N\geq 5$, setting

$$\beta=\alpha_1\alpha_2\alpha_3\cdots\alpha_{N-1}, \tag{24}$$

we have, from equation (19),

$$\alpha_i\beta = \beta\alpha_{i-2}, \quad N-1 \geq i \geq 3. \tag{25}$$

Then α_i can be expressed as

$$\alpha_i = \beta^{[\frac{i-1}{2}]}\alpha_{i-2[\frac{i-1}{2}]}\beta^{-[\frac{i-1}{2}]} \tag{26}$$

where $[\frac{i-1}{2}]$ is an integer but $\frac{i}{2}-1 \leq [\frac{i-1}{2}] \leq \frac{i-1}{2}$. So braid group $\hat{\mathrm{B}}_N(N \geq 5)$ has six generators: $\alpha_1, \alpha_2, \beta$ and the inverses of them: $\alpha_1^{-1}, \alpha_2^{-1}$ and β^{-1} due to

$$\alpha_{i-2[\frac{i-1}{2}]} = \begin{cases} \alpha_1 & \text{for odd } i \\ \alpha_2 & \text{for even } i \end{cases}. \tag{27}$$

The equation (17), that is, Fig.3, is the braid relation for the Frenkel-Moore equation which been written as [14]

$$S_{123}S_{214}S_{143}S_{234} = S_{234}S_{143}S_{214}S_{123}. \tag{28}$$

When we denote the indexes of S by the place where the strings cross, we have

$$S_{123}S_{234}S_{123}S_{234} = S_{234}S_{123}S_{234}S_{123}. \tag{29}$$

This is just the planar tetrahedral equation[3].

4 Transformations R and I

From relations (18) and (19) we get

$$S_{123}S_{145}S_{325}S_{234} = S_{234}S_{325}S_{145}S_{123}, \tag{30}$$

$$S_{123}S_{145}S_{254}S_{345} = S_{345}S_{254}S_{145}S_{123}. \tag{31}$$

The other two equations corresponding to relations (18) and (19) can be obtained from the above relations by using the index maps: $1 \longleftrightarrow 5$ and $2 \longleftrightarrow 4$. This means that some relations exist between the two equations in (17) or in (18), respectively. So we set

$$\begin{array}{ll} R(\alpha_i) = \alpha_{\sigma(i)}, & R(\alpha_i\alpha_j) = \alpha_{\sigma(i)}\alpha_{\sigma(j)}, \\ R(\alpha_i\alpha_j\alpha_k) = \alpha_{\sigma(i)}\alpha_{\sigma(j)}\alpha_{\sigma(k)}, & \cdots; \end{array} \tag{32}$$

$$
\begin{array}{ll}
I(\alpha_i)=\alpha_i, & I(\alpha_i\alpha_j)=\alpha_j\alpha_i, \\
I(\alpha_i\alpha_j\alpha_k)=\alpha_k\alpha_j\alpha_i, & \cdots.
\end{array} \tag{33}
$$

where $\sigma(i) = N - i$ and N is the total numbers of strings minus one. It can be proved easily that

$$
[R, I] = 0, \quad R^2 = I^2 = 1. \tag{34}
$$

The transformation R acting on the braids denotes the rotation of the braids through the angle π around the axis which the strings cross along. The transformation I acting on the braids denotes that the braids are rotated through the angle π about the axis which is in the plan which the braids are in and is perpendicular to the above axis. Then transformation RI describes the rotation of the braids through the angle π for the axis which is perpendicular to the plan in which the braids are. So the two relations in (18) or (19) can be changed each other by using the transformation I and (then) we can choice only one of them respectively. Acting on the generators: $\Delta = b_1 b_2 \cdots b_{N-1}$ and $\Omega = b_1\Delta$, of the ordinary braid group B_N by the transformations R and I, we have that

$$
\begin{array}{l}
R(\Delta)=I(\Delta)=\Delta^{N-1}(\Delta^{-1}\Omega\Delta^{-1})^{N-1}, \\
R(\Omega)=\Delta^{N-2}\Omega(\Delta^{-1}\Omega\Delta^{-1})^{N-1}, \\
I(\Omega)=\Delta^{N-1}(\Delta^{-1}\Omega\Delta^{-1})^{N-1}\Omega\Delta^{-1}.
\end{array} \tag{35}
$$

It is note that the braid relations of the planar permutohedron equation can be gotten easily from elements of B_N and $\hat{\mathrm{B}}_N$.

5 Conclusions and Remarks

Similarly as the above discussion, we can wright down the weight function of the three-dimensional vertex model related the 3D lattice integrable model proposed in refs. [11, 12] as the following form:

$$
R(u_1,u_2,u_3)_{i_1i_2i_3}^{j_1j_2j_3} = (-)^{j_2}(\omega^{1/2})^{j_1j_2+j_2j_3+j_1j_3}
$$

$$
\times\left[\frac{w(qu_1,j_1)w(q^{-1}(\omega u_2)^{-1},-i_2)w(q^{-1}u_3,i_3)}{w(q^{-1}u_1,i_1)w(q(\omega u_2)^{-1},-j_2)w(qu_3,j_3)}\right]^{1/2}\times
$$

$$\times \left\{ \sum_{\sigma \in Z_N} \frac{w(\omega u_2' u_3, \sigma + j_2 + j_3) w(q u_2'', \sigma) s(\sigma, j_1)}{w(q^{-1} u_2', \sigma + j_2) w(u_2'' u_3, \sigma + i_3)} \right\}_0 \tag{36}$$

It satisfies the modified tetrahedron equation

$$R_{123} \overline{R}_{145} R_{246} \overline{R}_{356} = R_{356} \overline{R}_{246} R_{145} \overline{R}_{123}, \tag{37}$$

where $\overline{R}(u_1, u_2, u_3)_{i_1 i_2 i_3}^{j_1 j_2 j_3}$ can be obtained from expression (36) by the substitutions:

$$q \to q^{-1}, \quad u_2' \to \overline{u}_2', \quad u_2'' \to \overline{u}_2'', \tag{38}$$

with the condition $u_2' \overline{u}_2'' = u_2'' \overline{u}_2'$. The details will be given else where. When $q = 1$, this vertex model reduces to the one in section 2. Then it can be regarded as a deformation of the three-dimensional vertex model corresponding to BB model.

As the conclusions we get the three-dimensional vertex model which is a duality of the three-dimensional Baxter-Bazhanov model. The Boltzmann weights of the model satisfy the vertex type tetrahedron equation. The braid group $\hat{\mathrm{B}}_N$ corresponding to the Frenkel-Moore equation is constructed and the transformations R and I acting on the braids and denoting the rotations of the braids around some special axes through the angle π are given. Using the method presented in this paper, we can constructed a new three-dimensional vertex model for which the weight function has the form (36). It is a duality of the 3D lattice integrable model proposed in refs. [11, 12] and a deformation of the vertex model related BB model. This means that we can interpreted the 3D lattice integrable model in ref. [11, 12] as a deformation of the three-dimensional Baxter-Bazhanov model. The symmetry properties of the weight function (36) can be given similar as in ref. [19].

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References

[1] Kashaev R M, Mangazeev V V and Stroganov Yu G 1993 *Int. J. Mod. Phys.* **A8** 1399.

[2] Frenkel I and Moore G 1991 *Commun. Math. Phys.* **138** 259.

[3] Carter J S and Saito M 1993 *Knotted surfaces, braid movies, and beyond*, July.

[4] Zamolodchikov A B 1981 *Commun. Math. Phys.* **79** 489.

[5] Hietarinta J 1994 *J. Phys.* **A27** 5727.

[6] Bazhanov V V and Baxter R J 1992 *J. Stat. Phys.* **69** 453.

[7] Kashaev R M, Mangazeev V V and Stroganov Yu G 1993 *Int. J. Mod. Phys.* **A8** 587.

[8] Hu Z N 1994 *Int. J. Mod. Phys.* **A9** 5201.

[9] Bazhanov V V and Baxter R J 1993 *J. Stat. Phys.* **71** 839.

[10] Hu Z N 1994 *Mod. Phys. Lett.* **B8** 779.

Hu Z N and Hou B Y 1995 *J. Stat. Phys.* **79** 759.

Hu Z N and Hou B Y 1995 *Cyclic quantum dilogarithm, shift operator and star-square relation of BB model*, *Phys. Lett.* **A** (to appear).

[11] Mangazeev V V, Sergeev S M and Stroganov Yu G 1994 *Int. J. Mod. Phys.* **A9** 5517.

[12] Boos H E, Mangazeev V V and Sergeev S M 1994 *Modified tetrahedron equations and related 3D integrable models*, June.

[13] Mangazeev V V and Stroganov Yu G 1993 *Mod. Phys. Lett.* **A8** 3475.

[14] Li Y Q and Hu Z N *Solutions of n-simplex equation from solutions of braid group representation*, *J. Phys.* **A** (to appear).

[15] Korepanov I G 1993 *Commun. Math. Phys.* **154** 85.

[16] Boukraa S, Maillard J M and Rollet G 1995 *J. Stat. Phys.* **78** 1195.

[17] Mangazeev V V, Sergeev S M and Stroganov Yu G 1994 *New solution of vertex type tetrahedron equations*, hep-th/9410049.

[18] Hu Z N 1995 *On the Three-dimensional Lattice Model*, AS-ITP-95-01.

[19] Hu Z N and Hou B Y 1995 *Three-dimensional vertex model in statistical mechanics, from Baxter-Bazhanov model*, hep-th/9503037.

Captions

1. Fig.1 α_i
2. Fig.2 α_i^{-1}
3. Fig.3 the graphic proof of $\alpha_i\alpha_{i+1}\alpha_i\alpha_{i+1} = \alpha_{i+1}\alpha_i\alpha_{i+1}\alpha_i$
4. Fig.4 the graphic proof of $\alpha_{i+1}\alpha_{i-1}\alpha_{i+1}\alpha_i = \alpha_i\alpha_{i-1}\alpha_{i+1}\alpha_{i-1}$
5. Fig.5 the graphic proof of $\alpha_{i+1}\alpha_{i-1}\alpha_i\alpha_{i+1} = \alpha_{i-1}\alpha_i\alpha_{i+1}\alpha_{i-1}$

Fig.1 α_i Fig.2 α_i^{-1}

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Fig.3 the graphic proof of $\alpha_i\alpha_{i+1}\alpha_i\alpha_{i+1} = \alpha_{i+1}\alpha_i\alpha_{i+1}\alpha_i$

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Fig.4 the graphic proof of $\alpha_{i+1}\alpha_{i-1}\alpha_{i+1}\alpha_i = \alpha_i\alpha_{i-1}\alpha_{i+1}\alpha_{i-1}$

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Fig.5 the graphic proof of $\alpha_{i+1}\alpha_{i-1}\alpha_i\alpha_{i+1} = \alpha_{i-1}\alpha_i\alpha_{i+1}\alpha_{i-1}$

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